INTRODUCTION TO OPTIMAL CONTROL

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Abstract. *This paper derives optimal control results from both the calculus of variations and dynamic programming. The proofs are self-contained and provide a compact reference for general results.*

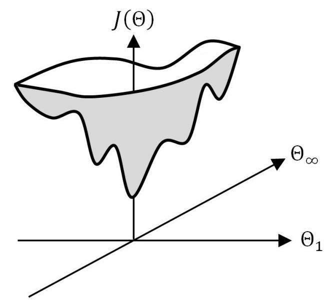

Figure 1. $J(\Theta)$ v.s. $\Theta \in \mathbb{R}^\infty$

1. Introduction

Consider the general dynamic ODE with time variable $t \in \mathbb{R}_{\geq 0}$, state variable $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$ with $m \leq n$. Assume the model $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ to be Lipchitz [3, p. 87] in x to ensure a well formed unique[1] solution. This deterministic system is given by

$$\dot{x}(t) = f(t, x(t), u). \tag{1}$$

Definition 1.1. Define the cost function such that $J : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$,

$$J(u) := \int_{t_1}^{t_2} \ell(t, x(t), u(t))dt. \tag{2}$$

The problem formulation is then to optimize (2) subject to (1) with specified boundary conditions. This paper introduces the first order necessary condition (FONC) of optimality. *This condition will produce a finite parametric collection of solutions from an infinite set of possible solutions.* A minimal or maximal solution can be characterized by an appropriate selection of ℓ, e.g., positive definite $\Rightarrow$ minimum, negative definite $\Rightarrow$ maximum.

As the problem is currently stated, this is an optimization problem over $n \times m$ continuous functions (each of which can be represented by an infinite number of parameters), e.g., $\Theta_{[j,:]} \in \mathbb{C}^\infty$, $j \in [1, n+m]$ with

$$x_j^*(t) = \sum_{k=-\infty}^{+\infty} \Theta_{j,k} e^{\mathbf{i}\left(\frac{2k\pi}{t_2 - t_1}\right)t} \qquad j \in [1, n],$$
$$u_j^*(t) = \sum_{k=-\infty}^{+\infty} \Theta_{j,k} e^{\mathbf{i}\left(\frac{2k\pi}{t_2 - t_1}\right)t} \qquad j \in [n+1, n+m].$$

Thus the problem can be restated as

$$\min_{\Theta \in \mathbb{C}^\infty} \quad J(\Theta) = \int_{t_1}^{t_2} \ell(t, x(t,\Theta), u(t,\Theta))dt$$
$$\text{s.t.} \quad \dot{x}(t,\Theta) = f(t, x(t,\Theta), u(t,\Theta)).$$

[1] Most natural phenomenon will satisfy this property locally.

Try picturing such a problem (figure 1). One possible solution, is to truncate the parametric representation to an approximate form, e.g., $\Theta_{[j,:]} \in \mathbb{R}^{K_j}$, $j \in [1, n+m]$, $K_j < \infty$ and

$$x_j^*(t,\Theta) = \sum_{k=0}^{K_j} \left(\frac{\Theta_{j,k}}{k!}\right) t^k, \qquad j \in [1, n],$$
$$u_j^*(t,\Theta) = \sum_{k=0}^{K_j} \left(\frac{\Theta_{i,k}}{k!}\right) t^k, \qquad j \in [n+1, n+m].$$

Such a solution is computationally expensive and inelegant. Can this problem be reformulated into a finite dimensional problem space? The answer is yes. In fact, there are several analytic ways to reformulate the problem, including the calculus of variations, dynamic programming and dual spaces [5, pg125]. More importantly, these methods provide insight into efficient numerical and iterative approaches that can solve problems that resist closed form solution.

2. The Calculus of Variations (CoV)

Consider the optimal curve $x^*(t)$ shown in figure 2 connecting the specified points (t_1, x_1) and (t_2, x_2). Now consider the family of perturbed curves $x(t,\Theta)$ parameterized by the scalar $\Theta \in \mathbb{R}$ with $x(t, 0) = x^*(t)$. This perturbed curve can be expanded at $\Theta = 0$ at a fixed time t with

$$x(t,\Theta) = \sum_{k=0}^{\infty} \frac{\Theta^k}{k!} \left[\frac{d^k}{d\Theta^k} x(t,\Theta)\right]_{\Theta=0}$$
$$= x^*(t) + \sum_{k=1}^{\infty} \frac{\Theta^k}{k!} \left[\frac{d^k}{d\Theta^k} x(t,\Theta)\right]_{\Theta=0}. \tag{3}$$

Definition 2.1. Define the variation operator

$$\delta^k(\,\cdot\,) := \frac{d^k}{d\Theta^k}(\,\cdot\,)\bigg|_{\Theta=0}. \tag{4}$$

Wherever δ is written without a k superscript it is assumed to mean δ^1. Also, note that $\delta^0 = 1$ (the identity operator). With this notation, equation (3) can be written more compactly.

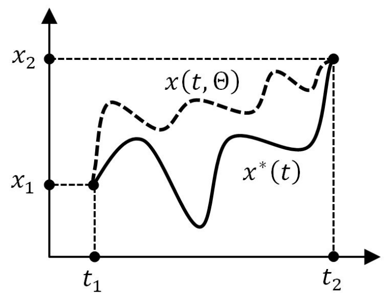

FIGURE 2. $x^*(t) = x(t,0)$ (the optimal trajectory) and $x(t,\Theta)$ (the sub-optimal trajectory) starting at specified initial time t_1 and state x_1 and ending at specified terminal time t_2 and state x_2

Definition 2.2. Define the perterbation residual

$$\tilde{x}(t,\Theta) = x(t,\Theta) - x^*(t). \tag{5}$$

The problem has reduced to

$$\begin{aligned} \min_{\Theta\in\mathbb{R}} \quad & J(\Theta) = \int_{t_1}^{t_2} \ell(t, x(t,\Theta), u(t,\Theta))dt \\ \text{s.t.} \quad & \dot{x}(t,\Theta) = f(t, x(t,\Theta), u(t,\Theta)), \end{aligned} \tag{6}$$

and by construction, the minimum is at $\Theta = 0$ (as seen in figure 3). The FONC is thus $\delta J = 0$ (Lemma 2.4) subject to the constraint (1).

Definition 2.3. Define the constraint equation

$$g(t,\Theta) := f(t,\Theta) - \dot{x}(t,\Theta) = 0.$$

If $\ell \succ 0$, (6) is equivalent to $\min \ell(\Theta)$ for fixed $t \in [t_1, t_2]$. Hence the integral can be initially dismissed from the optimization. For any fixed time t, this is a *Lagrange* multiplier problem (Lemma 2.5) in Θ. Therefore the constraint equation $g(t,\Theta)$ can be combined with (6) (at fixed time t) to get the augmented Lagrange equation $\ell(t,\Theta) + \lambda^T(t)g(t,\Theta)$. The FONC becomes

$$\begin{aligned} \delta J(\Theta) &= \delta \int_{t_1}^{t_2} \ell(t,\Theta) + \lambda^T(t) g(t,\Theta) dt \\ &= \int_{t_1}^{t_2} \delta\ell(t, x(t,\Theta), u(t,\Theta)) \\ &\quad + \delta\left(\lambda^T(t)\left(f\left(t, x(t,\Theta), u(t,\Theta)\right)\right) - \dot{x}(t,\Theta)\right) dt \\ &= \int_{t_1}^{t_2} \left(\delta x^T \frac{\partial \ell}{\partial x} + \delta u^T \frac{\partial \ell}{\partial u}\right. \\ &\quad \left. + \lambda^T \frac{\partial f}{\partial x}\delta x + \lambda^T \frac{\partial f}{\partial u}\delta u - \lambda^T \delta\dot{x}\right) dt \end{aligned} \tag{7}$$

For clarity, the variation of $\dot{x}$ is

$$\begin{aligned} \delta\dot{x} &:= \left[\frac{d}{d\Theta}\left(\frac{d}{dt}x(t,\Theta)\right)\right]_{\Theta=0} \\ &= \frac{d}{dt}\left[\frac{d}{d\Theta}x(t,\Theta)\right]_{\Theta=0} \\ &= \frac{d}{dt}\delta x. \end{aligned}$$

Now $\lambda^T\delta\dot{x}$ can be rewritten with the chain rule

$$\frac{d}{dt}\lambda^T\delta x = \dot{\lambda}^T\delta x + \lambda^T\frac{d}{dt}\delta\dot{x}.$$

Putting this back under the integral gives

$$\int_{t_1}^{t_2} -\lambda^T\delta\dot{x}dt = \int_{t_1}^{t_2} \dot{\lambda}^T\delta x dt - \left[\lambda^T(t)\delta x(t)\right]_{t_1}^{t_2} \tag{8}$$

Looking at (3), the only way that $x(t_1,\Theta)$ and $x(t_2,\Theta)$ can be equal to $x^*(t_1)$ and $x^*(t_2)$ for all possible Θ is

$$\delta^k x(t_1,\Theta) = \delta^k x(t_2,\Theta) = 0 \qquad \forall k > 0. \tag{9}$$

Combining (8) and (9) gives

$$\int_{t_1}^{t_2} -\lambda^T\delta\dot{x}dt = \int_{t_1}^{t_2} \dot{\lambda}^T\delta x dt,$$

which when combined with (7) gives

$$\begin{aligned} \delta J(\Theta) &= \int_{t_1}^{t_2} \delta x^T\left(\frac{\partial}{\partial x}\left(\ell + \lambda^T f\right) + \dot{\lambda}^T\right) dt \\ &\quad + \int_{t_1}^{t_2} \delta u^T \frac{\partial}{\partial u}\left(\ell + \lambda^T f\right) dt \end{aligned} \tag{10}$$

The perturbations δx and δu are both arbitrary functions of time. Therefore the terms multiplying them must be *identically* equal to zero, i.e. equal to zero over the interval $[t_1, t_2]$.

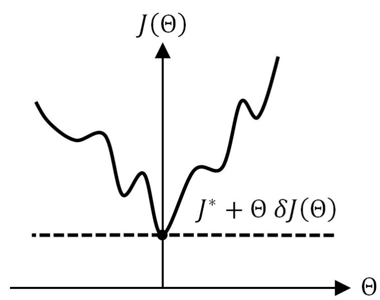

FIGURE 3. FONC for $\Theta \in \mathbb{R}$ is $\delta J = 0$

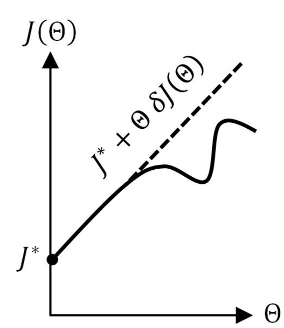

FIGURE 4. FONC for $\Theta \in \mathbb{R}_{\geq 0}$ is $\delta J \geq 0$

Lemma 2.4.
The FONC *comes from the Taylor expansion of* $J(\Theta)$, *which is given by*

$$J(\Theta) = J(0) + \Theta\delta J + \sum_{k=2}^{\infty}\left(\frac{\delta^k J}{k!}\right)\Theta^k,$$

where $J(0) = J^*$. *For* J^* *to be a minimum, it must be true that*

$$\begin{aligned} J^* &\leq J(\Theta) \\ &= J^* + \Theta\delta J + \sum_{k=2}^{\infty}\left(\frac{\delta^k J}{k!}\right)\Theta^k, \qquad \forall\Theta \neq 0. \end{aligned}$$

Subtract J^* *from both sides, divide by* $|\Theta|$, *and take the limit of* Θ *goes to zero to get*

$$\begin{aligned} \mathrm{sign}(\Theta)\delta J &\geq -\lim_{\Theta\to 0}\left\{|\Theta|^{-1}\sum_{k=2}^{\infty}\left(\frac{\delta^k J}{k!}\right)\Theta^k\right\} \\ &= -\lim_{\Theta\to 0}\left\{\mathrm{sign}(\Theta)\sum_{k=1}^{\infty}\left(\frac{\delta^{(k+1)} J}{(k+1)!}\right)\Theta^k\right\} \qquad = 0. \end{aligned}$$

If Θ *can be positive or negative,* $\delta J = 0$. *If the perturbation does not allow* Θ *to be both positive and negative, the inequality,* $\mathrm{sign}(\Theta)\delta J \geq 0$, *remains. This condition will be encountered when bounds are placed on* x *and* u. *Figure* 4 *depicts the positive case.*

Lemma 2.5.
Consider the optimization problem

$$\min_{\Theta} \quad \ell(\Theta) \quad \text{s.t.} \quad g(\Theta) = 0.$$

This is equivalent to (figure 5)

$$\begin{aligned} \min_{\Theta} \quad & H(\Theta) = \ell(\Theta) + \lambda^T g(\Theta) \\ \text{s.t.} \quad & g(\Theta) = 0, \end{aligned}$$

which is the familiar Lagrange multiplier problem. If the optimum value occurs at $\Theta = 0$ *and* f *and* g *are both differentiable in* Θ, *the problem further reduces to*

$$\begin{aligned} \delta(\ell(\Theta) + \lambda^T g(\Theta)) &= 0 \\ g(0) &= 0. \end{aligned} \tag{11}$$

Equation (11) *requires a regularity condition that* $\delta\ell$ *can be constructed from a linear combination of* δg.

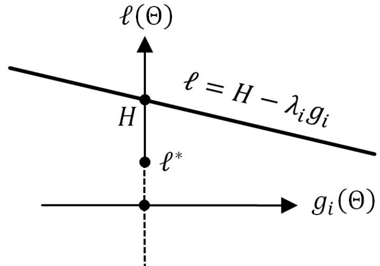

FIGURE 5. Minimizing $\ell(\Theta)$ with constraint $g(\Theta) = 0$ is dual to minimizing H with $g(\Theta) = 0$

3. Main Result: The Minimum Principle

Definition 3.1. Define the Hamiltonian

$$H := \ell + \lambda^T f, \qquad \ell \succ 0 \qquad \lambda \in \mathbb{R}^n. \tag{12}$$

Equation (10) becomes

$$\delta J(\Theta) = \int_{t_1}^{t_2}\left(\delta x^T\left(\frac{\partial H}{\partial x} + \dot{\lambda}\right) + \delta u^T\frac{\partial H}{\partial u}\right)dt$$

Definition 3.2. Define

$$H^*(t) := H(t, x^*(t), \lambda^*(t), u^*(t)). \tag{13}$$

To obtain $\delta J = 0$, the argument inside the integral must be identically equal to zero. The term factored in front of the δx term vanishes with the selection of

$$\dot{\lambda}^*(t) = -\left.\frac{\partial H}{\partial x}\right|_*, \tag{14}$$

where $|_*$ denotes evaluation at optimal values. Note

$$\dot{x}^*(t) = +\left.\frac{\partial H}{\partial \lambda}\right|_*. \tag{15}$$

The term in front of δu vanishes everywhere H is continuous and optimal in u thus motivating the claim

$$u^*(t) = \operatorname*{argmin}_u H\left(t, x^*(t), \lambda^*(t), u\right). \tag{16}$$

See [4, chp4] for detailed proof.

4. Free Boundary Conditions

The boundary conditions in the previous formulation were specified by $x^*(t_1) = x_1$ and $x^*(t_2) = x_2$. Now consider unspecified boundary conditions (depicted in figure 6). Extension of the main result above is very straight forward. Return to equation (8). Now for every value $x_i(t_1)$ and $x_j(t_2)$ that is not specified, it is no longer true that $\delta x_i(t_1)$ and $\delta x_j(t_2)$ are zero. In fact, they can now take on arbitrary values. Thus the first variation becomes

$$\begin{aligned} \delta J(\Theta) = &\int_{t_1}^{t_2}\left(\delta x^T\left(\frac{\partial H}{\partial x} + \dot{\lambda}\right) + \delta u^T\frac{\partial H}{\partial u}\right)dt \\ &-\left(\sum_j \lambda_j^T(t_2)\delta x_j(t_2) - \sum_i \lambda_i^T(t_1)\delta x_i(t_1)\right), \end{aligned}$$

and the corresponding value of $\lambda_i(t_1)$ or $\lambda_j(t_2)$ must be

$$\lambda_i(t_1) = 0, \qquad \lambda_j(t_2) = 0. \tag{17}$$

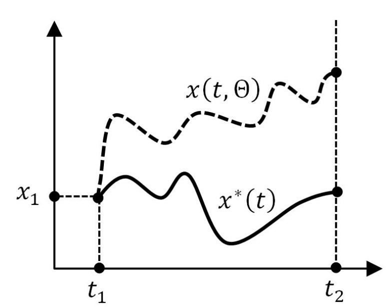

FIGURE 6. Free terminal boundary condition

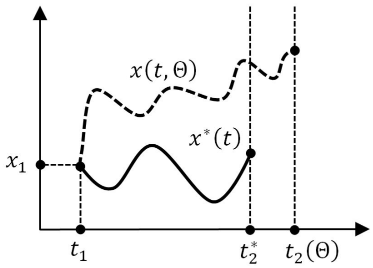

FIGURE 7. Free terminal time

5. Free Initial & Terminal Time

This scenario is depicted in figure 7. Perturbed temporal limits are given by

$$\tilde{t}_i(\Theta) := t_i(\Theta) - t_i^* = \sum_{k=1}^{\infty}\left(\frac{\delta^k t_i(\Theta)}{k!}\right)\Theta^k. \tag{18}$$

Returning to equation (7), and using the definition of H, the perturbed temporal model becomes

$$\begin{aligned}
\delta J &= \delta\left(\int_{t_1(\Theta)}^{t_2(\Theta)} H(t,\Theta)dt\right)\\
&= \delta\left(\int_{t_1^*}^{t_2^*} H(t,\Theta)dt\right) + \delta\left(\int_{t_2^*}^{t_2^*+\tilde{t}_2(\Theta)} H(t,\Theta)dt\right)\\
&\qquad - \delta\left(\int_{t_1^*}^{t_1^*+\tilde{t}_1(\Theta)} H(t,\Theta)dt\right)\\
&\approx \delta\left(\int_{t_1^*}^{t_2^*} H(t,\Theta)dt\right)\\
&\quad + \delta\left(\tilde{t}_2(\Theta)H\left(t_2^*,\Theta\right)\right) - \delta\left(\tilde{t}_1(\Theta)H\left(t_1^*,\Theta\right)\right)\\
&= \delta\left(\int_{t_1^*}^{t_2^*} H(t,\Theta)dt\right) + \delta t_2 H^*(t_2^*) - \delta t_1 H^*(t_1^*)
\end{aligned} \tag{19}$$

where

$$\delta\left(\tilde{t}_i(\Theta)H\left(t_i^*,\Theta\right)\right) = H\left(t_i^*,0\right)\delta\tilde{t}_i(\Theta) + \tilde{t}_i(0)\delta H\left(t_i^*,\Theta\right).$$

Here $\tilde{t}_i(0) = 0$ by construction. Note that $\delta\tilde{t}_i = \delta t_i$. The variation δt_i can take on arbitrary values. Therefore,

$$H^*\left(t_i^*\right) = 0 \qquad i \in \{1,2\}. \tag{20}$$

The sign in equation (19) will be important in the following sections.

6. Initial & Terminal Cost

Now consider adding a terminal cost to the free initial and terminal time problem,

$$\begin{aligned}
J(\Theta) &= \int_{t_1(\Theta)}^{t_2(\Theta)} H(t, x(t,\Theta), u(t,\Theta))dt\\
&\quad c\left(t_1(\Theta), x(t_1(\Theta),\Theta), t_2(\Theta), x(t_2(\Theta),\Theta)\right).
\end{aligned} \tag{21}$$

The variation on this terminal cost will be

$$\begin{aligned}
\delta c = \Bigg[&\delta t_1\frac{\partial c}{\partial t_1} + \delta x^T(t_1)\frac{\partial c}{\partial x(t_1)}\\
&+ \delta t_2\frac{\partial c}{\partial t_2} + \delta x^T(t_2)\frac{\partial c}{\partial x(t_2)}\Bigg]_*,
\end{aligned}$$

where $|_*$ denotes evaluation at optimal boundary values $(t_1^*, x^*(t_1^*), t_2^*, x^*(t_2^*))$. Again $\delta x_i(t_1^*)$ and $\delta x_j(t_2^*)$ takes on arbitrary values whenever $x_i(t_1^*)$ and $x_j(t_2^*)$ are not specified. Thus the first variation is given by

$$\begin{aligned}
\delta J(\Theta) = &\int_{t_1}^{t_2}\left(\delta x^T\left(\frac{\partial H}{\partial x} + \dot{\lambda}\right) + \delta u^T\frac{\partial H}{\partial u}\right)dt\\
&+ \delta t_1\left[\frac{\partial c}{\partial t_1} - H(t_1)\right]_*\\
&+ \delta t_2\left[\frac{\partial c}{\partial t_2} + H(t_2)\right]_*\\
&+ \sum_i \delta x_i^T(t_1^*)\left[\frac{\partial c}{\partial x_i(t_1)} + \lambda_i(t_1)\right]_*\\
&+ \sum_j \delta x_j^T(t_2^*)\left[\frac{\partial c}{\partial x_j(t_2)} - \lambda_j(t_2)\right]_*.
\end{aligned} \tag{22}$$

Therefore, the boundary conditions of the FONC are

$$0 = \left[\frac{\partial c}{\partial t_1} - H(t_1)\right]_* \qquad \text{if } t_1 \text{ not given,} \tag{23}$$

$$0 = \left[\frac{\partial c}{\partial t_2} + H(t_2)\right]_* \qquad \text{if } t_2 \text{ not given,} \tag{24}$$

$$0 = \left[\frac{\partial c}{\partial x_i(t_1)} + \lambda_i(t_1)\right]_* \qquad \text{if } x_i(t_1) \text{ not given,} \tag{25}$$

$$0 = \left[\frac{\partial c}{\partial x_j(t_2)} - \lambda_j(t_2)\right]_* \qquad \text{if } x_j(t_2) \text{ not given.} \tag{26}$$

7. Target Sets

This scenario is depicted in figure 8. Constrained target sets will require the use of additional Lagrange multipliers. Consider the general constraint equation

$$\gamma(t_1(\Theta), x(t_1,\Theta), t_2(\Theta), x(t_2(\Theta),\Theta)) = 0,$$

with $\gamma \in \mathbb{R}^p$. The constraints are added to the terminal cost with Lagrange multipliers of $\nu \in \mathbb{R}^p$. The cost

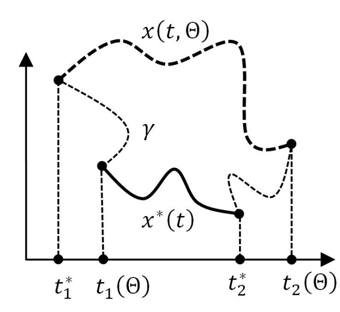

FIGURE 8. Target set

function given by equation (21) becomes

$$J(\Theta) = \int_{t_1(\Theta)}^{t_2(\Theta)} H(\Theta)dt + \Big(c(\Theta) + \nu^T\gamma(\Theta)\Big). \tag{27}$$

The first variation of γ is given by

$$\delta\gamma = \left[\delta t_2 \frac{\partial\gamma}{\partial t_2} + \delta x^T(t_1)\frac{\partial\gamma}{\partial x(t_1)} + \delta t_2 \frac{\partial\gamma}{\partial t_2} + \delta x^T(t_2)\frac{\partial\gamma}{\partial x(t_2)}\right]_*,$$

where $|_*$ denotes evaluation at optimal boundary values $(t_1^*, x^*(t_1), t_2^*, x^*(t_2^*))$. The constraint equations should be linearly independent, and the number of constraint equations plus the number of specified boundary values cannot exceed $2(n+1)$. If they do exceed this, the specified boundary values must be consistent with the constraint set, and constraints must be eliminated until only $2(n+1)$ remain. There is also a regularity condition that $\delta\gamma$ produces p linearly independent equations.

8. Summary of Results

With

- $H = \ell - \lambda^T f$,
- unspecified values of $x_i(t_1)$ and $x_j(t_2)$,
- variable times t_1 and t_2,
- terminal penalty c,
- and boundary constraints γ,

the first variation is given by

$$\begin{aligned}\delta J(\Theta) = &\int_{t_1}^{t_2}\left(\delta x^T\left(\frac{\partial H}{\partial x} + \dot\lambda\right) + \delta u^T\frac{\partial H}{\partial u}\right)dt\\ &+ \delta t_1\left[\frac{\partial}{\partial t_1}\Big(c + \nu^T\gamma\Big) - H(t_1)\right]_*\\ &+ \delta t_2\left[\frac{\partial}{\partial t_2}\Big(c + \nu^T\gamma\Big) + H(t_2)\right]_*\\ &+ \sum_i \delta x_i^T(t_1^*)\left[\frac{\partial}{\partial x_i(t_1)}\Big(c + \nu^T\gamma\Big) + \lambda_i(t_1)\right]_*\\ &+ \sum_j \delta x_j^T(t_2^*)\left[\frac{\partial}{\partial x_j(t_2)}\Big(c + \nu^T\gamma\Big) - \lambda_j(t_2)\right]_*.\end{aligned}$$

Therefore with $\lambda^*(t)$ given by (14), $x^*(t)$ given by (15), and $u^*(t)$ given by (16), the optimal boundary conditions become

$$0 = \left[\frac{\partial\left(c + \nu^T\gamma\right)}{\partial t_1} - H(t_1)\right]_* \quad \text{if } t_1 \text{ not given,} \tag{28}$$

$$0 = \left[\frac{\partial\left(c + \nu^T\gamma\right)}{\partial t_2} + H(t_2)\right]_* \quad \text{if } t_2 \text{ not given,} \tag{29}$$

$$0 = \left[\frac{\partial\left(c + \nu^T\gamma\right)}{\partial x_i(t_1)} + \lambda_i(t_1)\right]_* \quad \text{if } x_i(t_1) \text{ not given,} \tag{30}$$

$$0 = \left[\frac{\partial\left(c + \nu^T\gamma\right)}{\partial x_j(t_2)} - \lambda_j(t_2)\right]_* \quad \text{if } x_j(t_2) \text{ not given,} \tag{31}$$

$$0 = \gamma(t_1^*, x^*(t_1^*), t_2^*, x^*(t_2^*)). \tag{32}$$

9. The Second Order Necessary Condition

It is possible to further reduce the set of candidate solutions by solving $\delta^2 J(\Theta) \geq 0$. In general, the second variation is harder to solve because it is no longer possible to integrate by parts. It is typically easier just to find all candidate solutions and plug them into equation (2) to determine which one is optimal. In the case that no optimal solution exists, it should be fairly easy to find a counter-example more optimal than the one given by the FONC. If H is twice differentiable in u, a necessary but not sufficient condition is

$$\left.\frac{\partial^2 H}{\partial u^2}\right|_* \geq 0. \tag{33}$$

10. The Euler-Lagrange Equation

The original problem formulation by Euler and Lagrange can be captured by the general formulation of (6). Indeed, it is just one specific form of many. The first order Euler-Lagrange Equation (ELE) is given by

$$J = \int_{t_1}^{t_2} \ell\left(t, y(t), \dot y(t)\right)dt, \qquad y \in \mathbb{R}^n.$$

Set $x(t) = y(t)$ and $u(t) = \dot y(t)$. The problem is reformulated with

$$\begin{aligned} f &:= u(t),\\ H &:= \ell(t, x(t), u(t)) + \lambda^T(t)u(t).\end{aligned}$$

This problem can now be solved with the methods outlined above. It is important to note that continuity of u is not required. So this reformulation will produce a stronger minimum than if only continuous $\dot y(t)$ are considered. It is also possible to solve the ELE for k piecewise discontinuities using the Weierstrass-Erdmann-corner conditions [4, pg71], sifting through $k \geq 1$ until a minimum is found. There is, however, no upper bound on k, and it is possible that there are a countably infinite number of discontinuities.

Definition 10.1. Define

$$y^{(k)}(t) := \frac{d^k}{dt^k} y(t).$$

By induction, the N^{th} order ELE with $x \in \mathbb{R}^{nN}$ and

$$J = \int_{t_1}^{t_2} \ell\left(t, \left\{y^{(k)}(t)\right\}_{k=0}^{N}\right)dt \qquad N > 1,$$

is reformulated with $u(t) = y^{(N)}(t)$ and

$$x(t) = \begin{pmatrix} y(t)\\ \dot y(t)\\ \ddot y(t)\\ \vdots\\ y^{(N-1)}(t)\end{pmatrix}.$$

This gives

$$
f(x,u)=\begin{pmatrix} 0 & I_n & & 0 \\ & & \ddots & \\ & & & I_n \\ 0 & & & 0 \end{pmatrix} x+\begin{pmatrix} 0 \\ \vdots \\ 0 \\ I_n \end{pmatrix} u. \tag{34}
$$

The solution can also be obtained directly through repeated integration by parts and induction. The first variation is given by

$$
\begin{aligned}
\delta J &= \int_{t_1}^{t_2}\left(\frac{\partial \ell}{\partial y}\delta y+\sum_{k=1}^{N}\left(\frac{\partial \ell}{\partial y^{(k)}}\right)\delta y^{(k)}\right)dt \\
&= \int_{t_1}^{t_2}\left(\frac{\partial \ell}{\partial y}-\frac{d}{dt}\frac{\partial \ell}{\partial \dot{y}}\right)\delta y dt \\
&\quad -\int_{t_1}^{t_2}\sum_{k=2}^{N}\left(\delta y^{(k-1)}\frac{d}{dt}\left(\frac{\partial \ell}{\partial y^{(k)}}\right)\right)dt \\
&\quad +\left[\frac{\partial \ell}{\partial \dot{y}}\delta y+\sum_{k=2}^{N}\left(\frac{\partial \ell}{\partial y^{(k)}}\right)\delta y^{(k-1)}\right]_{t_1}^{t_2} \\
&\quad \vdots \\
&= \int_{t_1}^{t_2}\left(\delta y\sum_{k=0}^{N}(-1)^k\frac{d^k}{dt^k}\left(\frac{\partial \ell}{\partial y^{(k)}}\right)\right)dt \\
&\quad +\left[\sum_{i=1}^{N}\left(\sum_{k=0}^{N-i}(-1)^k\frac{d^k}{dt^k}\left(\frac{\partial \ell}{\partial y^{(k+i)}}\right)\right)\delta y^{(i-1)}\right]_{t_1}^{t_2}
\end{aligned}
$$

With this approach, the FONC is found to be

$$
\sum_{k=0}^{N}(-1)^k\frac{d^k}{dt^k}\left(\frac{\partial \ell}{\partial y^{(k)}}\right)=0 \qquad \forall t\in[t_1,t_2], \tag{35}
$$

with

$$
\sum_{k=0}^{N-i}(-1)^k\frac{d^k}{dt^k}\left(\frac{\partial \ell}{\partial y^{(k+i)}}\right)\Bigg|_{t=t_j}=0. \tag{36}
$$

for each $y^{(i-1)}(t_j)$ not specified, $i\in[1,N]$ and $j\in\{1,2\}$. The SONC is given by Legendre's condition,

$$
\frac{\partial^2 \ell}{\left(\partial y^{(N)}\right)^2}\geq 0. \tag{37}
$$

See [4, pg58] for details. With the reformulation given by equation (34), $y^{(N)}\to u$ and $\ell\to H$, and the SONC matches equation (33). A sufficiency condition can be derived from the Jacobi equation [4, pg65].

11. Reference Tracking

Extension to reference tracking problems is a trivial matter of variable transformation. The problem formulation is

$$
\begin{aligned}
\min_{\Theta\in\mathbb{R}} \quad & J(\Theta)=\int_{t_1}^{t_2}\ell(t,e(t,\Theta),u(t,\Theta))dt \\
\text{s.t.} \quad & \dot{x}(t,\Theta)=f(t,x(t,\Theta),u(t,\Theta)) \\
& e(t):=x(t)-r(t).
\end{aligned}
$$

Expressing the constraint just in terms of e gives

$$
\dot{e}(t,\Theta)=f(t,r(t)-e(t)(t,\Theta),u(t,\Theta))-\dot{r}(t).
$$

12. Dynamic Programming (DP)

Principle of DP.
All sub-paths of an optimal path are optimal.

The DP problem is depicted in figure 9 for some fixed time t_1 and t_2 and a time $t_0\in(t_1,t_2)$ with $x^*(t_0)=x_0$. Optimality is given with

$$
\begin{aligned}
J^*(t_0,x_0)=&\min_u\left\{\int_{t_0}^{t_2}\ell\left(t,x^*(t),u(t)\right)dt\right\}, \\
&\text{s.t.} \quad \dot{x}^*(t)=f\left(t,x^*(t),u(t)\right), \quad x^*(t_0)=x_0.
\end{aligned}
$$

Definition 12.1. Define

$$
\ell^*(t):=\ell\left(t,x^*(t),u^*(t)\right).
$$

Select a sub-path $[t_0,t_0+\Theta)$ on $t\in[t_0,t_2]$,

$$
\begin{aligned}
J^*(t_0,x_0) &= \int_{t_0}^{t_2}\ell^*(t)dt \\
&= \int_{t_0}^{t_0+\Theta}\ell^*(t)dt+\int_{t_0+\Theta}^{t_2}\ell^*(t)dt.
\end{aligned} \tag{38}
$$

For small Θ the first term becomes

$$
\int_{t_0^*}^{t_0+\Theta}\ell^*(t)dt\approx\ell^*(t_0)\Theta.
$$

The second term is

$$
\int_{t_0+\Theta}^{t_2}\ell^*(t)dt=J^*\left(t_0+\Theta,x^*(t_0+\Theta)\right).
$$

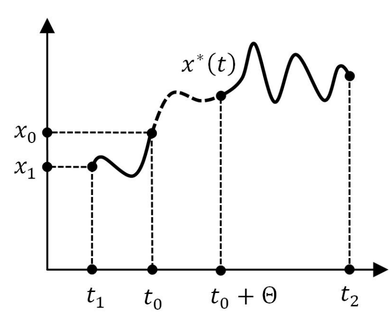

Figure 9. Dynamic Programming

The Taylor expansion of this term is

$$J^*\left(t_0+\Theta, x^*\left(t_0+\Theta\right)\right) \approx J^*\left(t_0, x_0\right)+\left.\frac{\partial J^*}{\partial t}\right|_{t_0, x_0} \Theta + \left(x^*\left(t_0+\Theta\right)-x^*\left(t_0\right)\right)^T \left.\frac{\partial J^*}{\partial x^*}\right|_{t_0, x_0} .$$

Putting this all together, equation (38) becomes

$$J^*\left(t_0, x_0\right) \approx \ell^*\left(t_0\right) \Theta+J^*\left(t_0, x_0\right)+\left.\frac{\partial J^*}{\partial t}\right|_{t_0, x_0} \Theta + \left(x^*\left(t_0+\Theta\right)-x^*\left(t_0\right)\right)^T \left.\frac{\partial J^*}{\partial x^*}\right|_{t_0, x_0} .$$

Subtracting $J^*\left(t_0, x_0\right)$ from both sides, dividing by Θ, and taking the limit as $\Theta \rightarrow 0$ gives

$$0=\ell^*\left(t_0\right)+\left.\frac{\partial J^*}{\partial t}\right|_{t_0, x_0} + \lim_{\Theta \rightarrow 0}\left(\frac{x^*\left(t_0+\Theta\right)-x^*\left(t_0\right)}{\Theta}\right)^T \left.\frac{\partial J^*}{\partial x^*}\right|_{t_0, x_0} .$$

The limit converges to $\dot{x}^*(t_0)=f^*(t_0)$. Thus

$$\begin{aligned}
-\left.\frac{\partial J^*}{\partial t}\right|_{t_0, x_0} &= \ell^*\left(t_0, x^*\left(t_0\right), u^*\left(t_0\right)\right) \\
&\quad + f^T\left(t_0, x^*\left(t_0\right), u^*\left(t_0\right)\right) \left.\frac{\partial J^*}{\partial x^*}\right|_{t_0, x_0} \\
&= \min_u \left.\left\{\ell(u)+f^T(u) \frac{\partial J^*}{\partial x^*}\right\}\right|_{t_0, x_0} .
\end{aligned}$$

Now, for any arbitrary initial time t_0 and x_0, this holds true. Thus

$$-\frac{\partial J^*}{\partial t}=\min_u\left\{\ell(u)+f^T(u) \frac{\partial J^*}{\partial x^*}\right\} \quad \forall t \in\left[t_1, t_2\right]. \tag{39}$$

The right hand side looks identical to the Hamiltonian found from CoV. Thus motivating the relationship

$$\lambda^*(t)=\frac{\partial J^*}{\partial x^*}. \tag{40}$$

With this relationship, equation (39) becomes

$$-\frac{\partial J^*}{\partial t}=\min_u H(u)=H^*. \tag{41}$$

Taking the partial in time of equation (40) gives

$$\dot{\lambda}^*(t)=\frac{\partial^2 J^*}{\partial x^* \partial t}=-\frac{\partial}{\partial x^*}\left(-\frac{\partial J^*}{\partial t}\right)=-\frac{\partial H^*}{\partial x^*}, \tag{42}$$

which gives the same result as equation (14) found with CoV. Solving (39) will give

$$u^*(t)=u^*\left(t, \frac{\partial J^*(t, x^*)}{\partial x^*}\right),$$

which has explicit dependence on x^*. This is the major tradeoff to the DP approach. The control u is now solved as a function of both the time and the state making it more robust to discrepancies between the model and real world applications. It comes at a computational cost however. The CoV approach produces ODEs while the DP approach produces PDEs. It is important to note that the optimal boundary conditions given in equations (28)–(32) still apply to the PDE form.

13. The Linear Quadratic Regulator (LQR)

The LQR is a specific DP solution to a commonly encountered class of problems. Consider the quadratic cost

$$\ell=\frac{1}{2} x^T Q x+\frac{1}{2} u^T R u$$

and the linear system

$$f=A x+B u$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $Q \succ 0$, and $R \succ 0$. The Hamiltonian is given by

$$H=\frac{1}{2} x^T Q x+\frac{1}{2} u^T R u+\lambda^T(A x+B u).$$

Equation (16) gives

$$H_{u^*}^*=R u^*+B^T \lambda^*=0,$$

which when solved gives

$$u^*=-R^{-1} B^T \lambda^*. \tag{43}$$

Suppose the solution is a linear state feedback control of the form $u^*=K^*(t) x^*$. Using the DP formulation, this would give a cost function of the form

$$J^*\left(t, x^*\right)=\frac{1}{2} x^{*T} P(t) x^* .$$

Thus λ^* would be

$$\lambda^*=\frac{\partial J^*}{\partial x^*}=P(t) x^*,$$

and

$$\dot{\lambda}^*=\frac{d}{d t} \frac{\partial J^*}{\partial x^*}=\dot{P}(t) x^*+P(t) \dot{x}^* . \tag{44}$$

Equation (14) gives

$$\dot{\lambda}^*=-\frac{\partial H^*}{\partial x^*}=-Q x^*-A^T \lambda^* . \tag{45}$$

Combining equations (45) and (44) gives

$$\begin{aligned}
-\dot{P}(t) x^* &= Q x+A^T \lambda+P(t) \dot{x}^* \\
&= Q x^*+A^T \lambda^*+P(t)\left(A x^*+B u^*\right) \\
&= Q x^*+A^T \lambda^*+P(t)\left(A x^*-B R^{-1} B^T \lambda^*\right) \\
&= Q x^*+A^T P(t) x^*+P(t) A x^* \\
&\qquad - P(t) B R^{-1} B^T P(t) x^*
\end{aligned}$$

This equation must hold for all x^*. Therefore

$$-\dot{P}(t)=Q+A^T P(t)+P(t) A-P(t) B R^{-1} B^T P(t). \tag{46}$$

Pluggin λ^* back into the control input gives

$$u^*=K^*(t) x^*, \qquad K^*(t)=-R^{-1} B^T P(t). \tag{47}$$

References

[1] Richard Bellman. *Dynamic Programming.* Dover, 2003.
[2] I. M. Gelfand and S. V. Fomin. *Calculus of Variations.* Dover, 2000.
[3] Hassan K. Khalil. *Nonlinear Systems.* Prentice Hall, 3 edition, 2002.
[4] Dan Liberzon. *Calculus of Variations and Optimal Control Theory: A Concise Introduction.* Princeton, 2012.
[5] David G. Luenberger. *Optimization by Vector Space Methods.* Wiley, 1969.